Math 155, ***Lecture Notes- Bonds*** **Name**____________

Section 9.9 Representation of Functions by Power Series

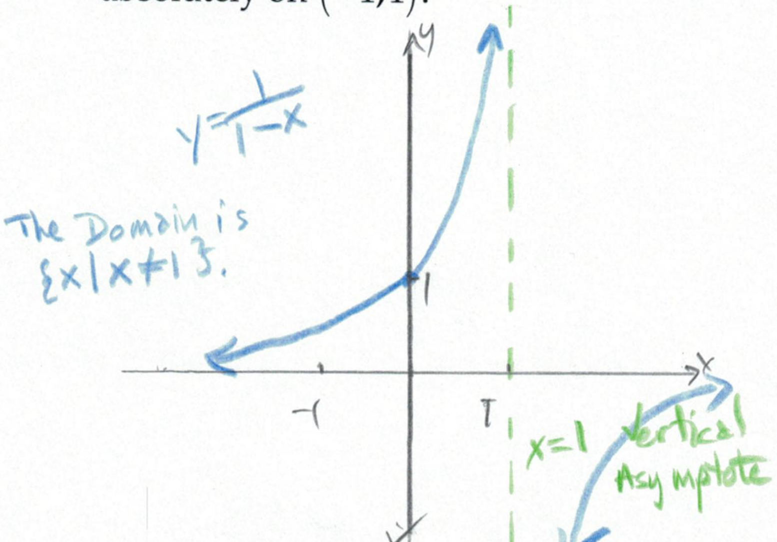
In this section, we will consider a few interesting techniques that will allow us to find a power series that represents a given function. In particular, we will focus on using the formula for the sum of a convergent geometric series to define a power series representation of a particular function. If needed we can move the center of the series, we can perform algebraic operations with a series, or combinations of series, or we can use calculus based operations like differentiation, or integration to create a particular series representation of a given function.

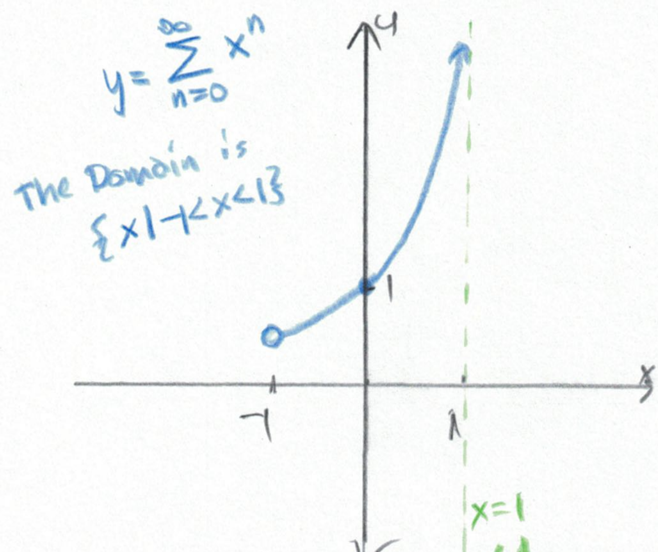
From Section 9.2, we can recall the following theorem:

> **THEOREM 9.6 Convergence of a Geometric Series**
>
> A geometric series with ratio r diverges if $|r| \ge 1$. If $0 < |r| < 1$, then the series converges to the sum
>
> $$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

Ex. 1: If we let $a=1$ and $r=x$, the geometric series sum formula gives us a power series representation for $f(x)=\dfrac{1}{1-x}$ centered at $c=0$.

That is, $f(x)=\dfrac{1}{1-x}=\sum_{n=0}^{\infty}(1)(x)^n=\sum_{n=0}^{\infty}x^n$, for $|x|<1$. This series converges absolutely on $(-1,1)$.





We will use this geometric power series sum formula to develop many other representations of functions by manipulating values of a, r, and c.

Ex. 2: Use the geometric series sum formula to represent $f(x)=\dfrac{1}{1-x}$ as a power series centered at $c=-1$, and find the domain of this power series function.

When we change the center of this power series, we should see $(x+1)^?$, which will show the new center at $c=-1$. Also, we will be able to find a corresponding change in the domain of the power series representation, since we will be moving the center of the previous interval of convergence, $(-1,1)$.

Let $z=x+1$, and we can see $z-1=x$. When we use $z=x+1$, we can "hold" the information that $c=-1$ is the new center of our power series representation. We can use $x=z-1$ to substitute for x in $f(x)$, and then we can use algebra to find $\frac{a}{1-r}=\sum_{n=0}^{\infty} ar^n$.

Consider $f(x)=\dfrac{1}{1-x}$

Try to manipulate this to look like $\dfrac{a}{1-r}$.

$$f(z-1)=\frac{1}{1-(z-1)}=\frac{1}{1-z+1}=\frac{1}{2-z}$$

$$=\left(\frac{1}{2-z}\right)\left[\frac{\frac{1}{2}}{\frac{1}{2}}\right]=\frac{\frac{1}{2}}{1-\frac{z}{2}}=\frac{a}{1-r}$$

$a=\frac{1}{2}$

$r=\frac{z}{2}$

$$=\sum_{n=0}^{\infty} ar^n$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{z}{2}\right)^n=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n(z)^n$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1}(z)^n$$

back substitute for $x+1=z$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1}(x+1)^n$$

$$f(x)=\sum_{n=0}^{\infty}\frac{(x+1)^n}{2^{n+1}}$$

Now, we can find the interval of convergence, which will be the Domain of this power series representation.

More Ex. 2:

According to Theorem 9.6, our geometric series converges when $|r|<1$. So, we can employ a few substitutions and solve the inequality for x, giving us an interval of convergence.

$$|r|<1$$

$$\left|\frac{z}{2}\right|<1$$

$$\left|\frac{x+1}{2}\right|<1 \quad \leftarrow \underline{\text{Solve for } x!}$$

$$2\cdot\left|\frac{x+1}{2}\right|<2\cdot 1$$

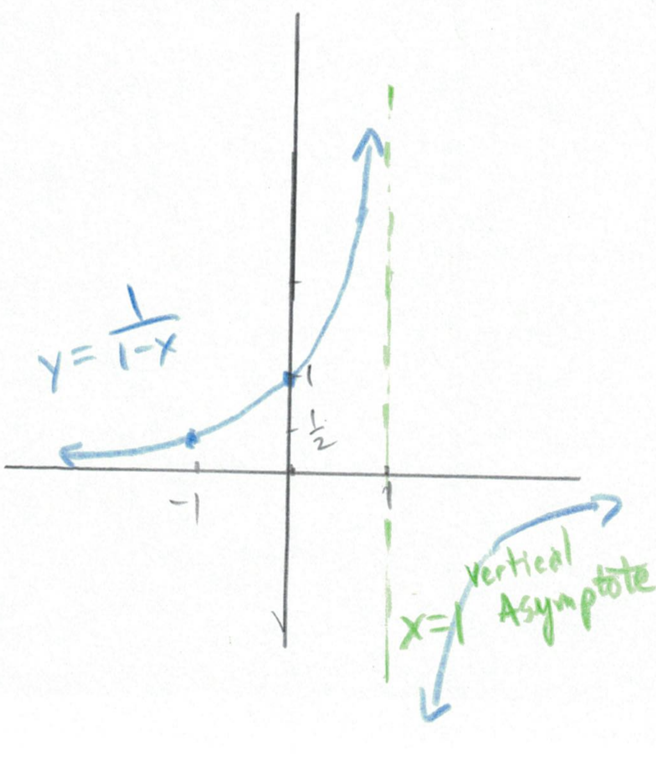
$$|x+1|<2$$

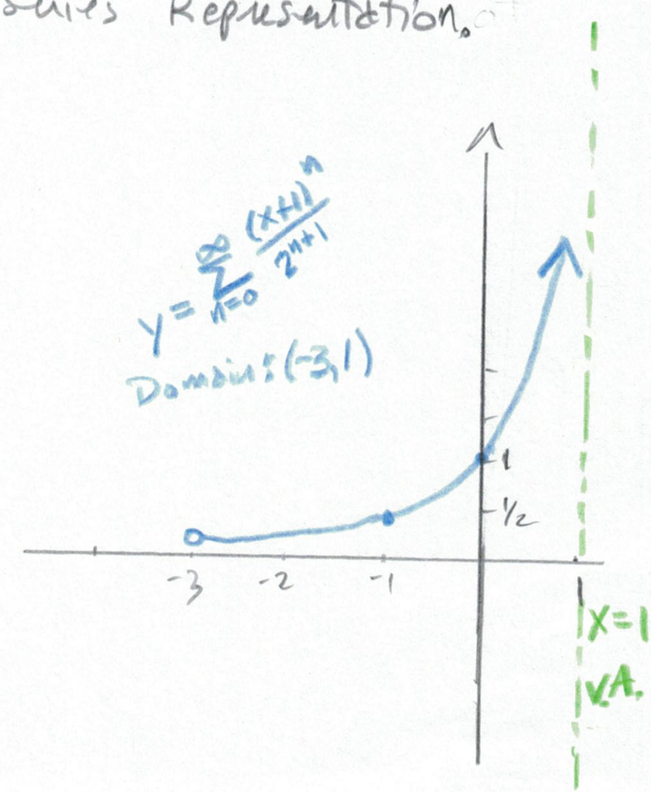
$$-2<x+1<2$$

$$-2-1<x+1-1<2-1$$

$$\underline{-3<x<1}$$

. So, our interval of convergence is $(-3,1)$, and this is the Domain of $\sum_{n=0}^{\infty}\frac{(x+1)^n}{2^{n+1}}$ as a Power Series Representation.





Ex. 3: Use the geometric series sum formula to represent $f(x) = \dfrac{4}{3x+2}$ as a power series centered at $c = 2$, and find the interval of convergence (domain) of this power series function.

Let $z = x-2$, and we can see $z+2 = x$. When we use $z = x-2$, we can "hold" the information that $c=2$ is the new center of our power series representation. We can use $x = z+2$ to substitute for x in $f(x)$, and then we can use algebra to find $\dfrac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$.

Try to manipulate this to look like $\dfrac{a}{1-r}$.

Consider $f(x) = \dfrac{4}{3x+2}$

$$f(z+2) = \frac{4}{3(z+2)+2} = \frac{4}{3z+6+2} = \frac{4}{3z+8} = \frac{4}{8+3z}$$

$$= \left[\frac{\frac{4}{1}}{\frac{8}{1}+\frac{3z}{1}}\right]\cdot\left[\frac{\frac{1}{8}}{\frac{1}{8}}\right] = \frac{\frac{4}{8}}{1+\frac{3z}{8}} = \frac{\frac{1}{2}}{1-\left(\frac{-3z}{8}\right)} = \frac{a}{1-r}$$

$a = \frac{1}{2}$
$r = \frac{-3z}{8}$

$$= \sum_{n=0}^{\infty} ar^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{-3z}{8}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-3}{8}\right)^n (z)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{8}\right)^n(x-2)^n$$

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{3(x-2)}{8}\right]^n$$

Now, we can find the interval of convergence, will be the Domain of this power series representation.

According to Theorem 9.6, our geometric series converges when $|r|<1$. So, we can employ a few substitutions and solve the inequality for x, giving us an interval of convergence.

More Ex. 3:

$$|r| < 1$$

$$\left|\frac{-3z}{8}\right| < 1$$

$$\left|\frac{-3(x-2)}{8}\right| < 1$$

$$\frac{8}{3}\cdot\left|\frac{3(x-2)}{8}\right| < \frac{8}{3}\cdot 1$$

$$|x-2| < \frac{8}{3}$$

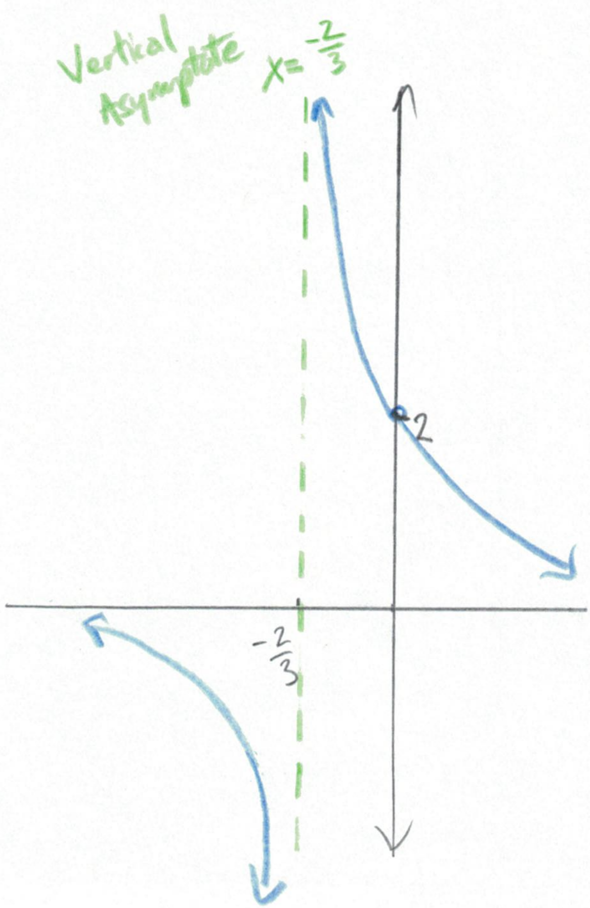
$$-\frac{8}{3} < x-2 < \frac{8}{3}$$

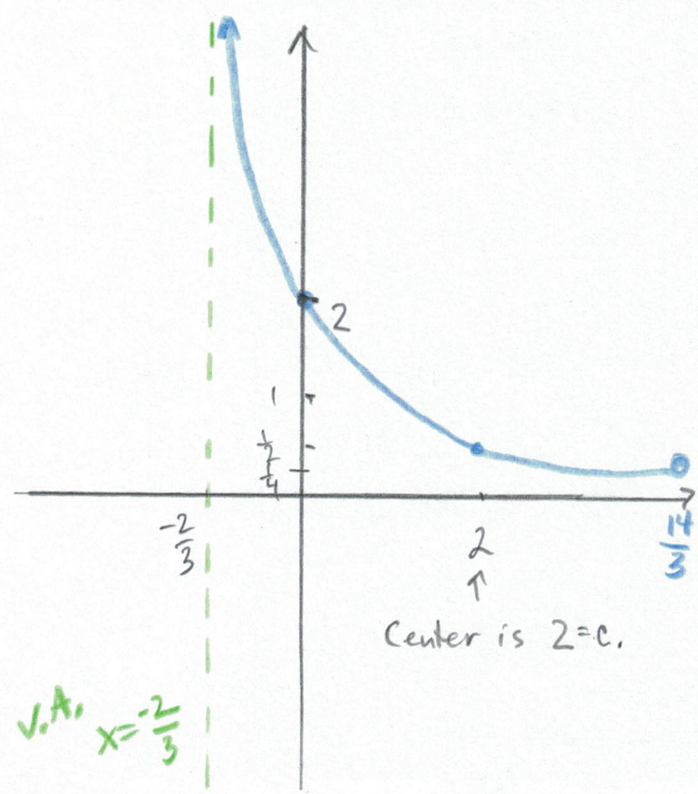
$$-\frac{8}{3} + 2 < x-2+2 < \frac{8}{3} + 2$$

$$-\frac{8}{3} + \frac{6}{3} < x < \frac{8}{3} + \frac{6}{3}$$

$$-\frac{2}{3} < x < \frac{14}{3}.$$

So, our interval of convergence is $\left(-\frac{2}{3}, \frac{14}{3}\right)$, and this is the Domain of $\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{3(x-2)}{8}\right]^n$ as a Power series Representation.





Operations with Power Series

Let $f(x) = \Sigma\, a_n x^n$ and $g(x) = \Sigma\, b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$

2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

NOTE:
- For simplicity, the properties are stated for series centered at $c = 0$.
- These operations can change the interval of convergence.
- When two series are summed, the interval of convergence for the sum is the intersection of the intervals of convergence of "the" two original series.

Ex. 4: Use the geometric series sum formula to represent $g(x) = \dfrac{4x-7}{2x^2+3x-2}$ as a power series centered at $c = 0$, and find the interval of convergence (domain) of this power series function. First, we should decompose $g(x) = \dfrac{4x-7}{2x^2+3x-2}$ into two simpler expressions.

We have $\dfrac{4x-7}{2x^2+3x-2} = \dfrac{4x-7}{(2x-1)(x+2)} = \dfrac{A}{x+2} + \dfrac{B}{2x-1}$

$4x-7 = A(2x-1) + B(x+2)$ ← Basic Equation, using Partial Fraction Decomposition.

More Ex. 4:

To Solve for A & B, we can let $x=-2$ and then let $x=\frac{1}{2}$.

Let $x=-2$, $4(-2)-7=A[2(-2)-1]+B[(-2)+2]$

$-8-7=A[-4-1]+B\cdot[0]$

$-15=-5A$

$\frac{-15}{-5}=A$

$\boxed{A=3}$

Let $x=\frac{1}{2}$, $4(\frac{1}{2})-7=3[2(\frac{1}{2})-1]+B[(\frac{1}{2})+2]$

$2-7=3[1-1]+B[\frac{1}{2}+\frac{4}{2}]$

$-5=3\cdot[0]+\frac{5}{2}B$

$-5=\frac{5}{2}B$

$-\frac{5}{1}\cdot\frac{2}{5}=B$

$\boxed{-2=B}$ So, $\boxed{g(x)=\frac{4x-7}{2x^2+3x-2}=\frac{3}{x+2}+\frac{-2}{2x-1}}$

Let's try to write each expression as a geometric power series.

$$\frac{a}{1-r}=\sum_{n=0}^{\infty}ar^n$$

$$\frac{3}{x+2}=\frac{3}{2+x}=\left[\frac{\frac{3}{1}}{\frac{2}{1}+\frac{x}{1}}\right]\cdot\left[\frac{\frac{1}{2}}{\frac{1}{2}}\right]=\frac{\frac{3}{2}}{1+\frac{x}{2}}=\frac{\frac{3}{2}}{1-(\frac{-x}{2})}=\frac{a_1}{1-r_1}\quad\boxed{\begin{aligned}a_1&=3/2\\ r_1&=\frac{-x}{2}\end{aligned}}$$

$$\frac{-2}{2x-1}=\frac{-2}{-1+2x}=\left[\frac{-2}{-1+2x}\right]\cdot\left[\frac{-1}{-1}\right]=\frac{2}{1-2x}=\frac{a_2}{1-r_2}\quad\boxed{\begin{aligned}a_2&=2\\ r_2&=2x\end{aligned}}$$

This means that $\frac{3}{x+2}=\frac{\frac{3}{2}}{1-(\frac{-x}{2})}=\sum_{n=0}^{\infty}a_1r_1^n=\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)\left(\frac{-x}{2}\right)^n=\sum_{n=0}^{\infty}\frac{3}{2}\cdot\frac{(-1)^nx^n}{2^n}$

$$=\sum_{n=0}^{\infty}\frac{3(-1)^nx^n}{2^{n+1}} ✓$$

$$\frac{2}{1-2x}=\sum_{n=0}^{\infty}a_2r_2^n=\sum_{n=0}^{\infty}(2)(2x)^n=\sum_{n=0}^{\infty}2\cdot2^nx^n=\sum_{n=0}^{\infty}2^{n+1}x^n$$

$$=\sum_{n=0}^{\infty}2^{n+1}x^n ✓$$

Still More Ex. 4:

We have $g(x) = \dfrac{4x-7}{2x^2+3x-2}$

$$g(x) = \frac{3}{x+2} + \frac{-2}{2x-1}$$

$$g(x) = \frac{3}{x+2} + \frac{2}{1-2x}$$

$$g(x) = \sum_{n=0}^{\infty} \frac{3(-1)^n x^n}{2^{n+1}} + \sum_{n=0}^{\infty} 2^{n+1} x^n$$

$$g(x) = \sum_{n=0}^{\infty} \left[\frac{3(-1)^n x^n}{2^{n+1}} + 2^{n+1} x^n\right]$$

$$\boxed{g(x) = \sum_{n=0}^{\infty} \left[\frac{3(-1)^n}{2^{n+1}} + 2^{n+1}\right] x^n}$$

To find the domain, or the interval of convergence for this series, we need to find the intersection of the intervals of convergence for each of the two original power series.

To find the interval of convergence for $\dfrac{3}{x+2} = \displaystyle\sum_{n=0}^{\infty} \frac{3(-1)^n x^n}{2^{n+1}}$

$$= \sum_{n=0}^{\infty} \frac{3}{2}\left(\frac{-x}{2}\right)^n$$

$= \displaystyle\sum_{n=0}^{\infty} a_1 r_1^n$, we can solve $|r_1| < 1$.

$$\left|\frac{-x}{2}\right| < 1$$

$$2 \cdot \left|\frac{x}{2}\right| < 2 \cdot 1$$

$$|x| < 2$$

$$-2 < x < 2$$

So, the interval of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2}\left(\frac{-x}{2}\right)^n$ is $(-2, 2)$. $R_1 = 2$

To find the interval of convergence for $\dfrac{2}{1-2x} = \displaystyle\sum_{n=0}^{\infty} 2^{n+1} x^n = \sum_{n=0}^{\infty} 2 \cdot 2^n x^n$

$$= \sum_{n=0}^{\infty} 2(2x)^n$$

$= \displaystyle\sum_{n=0}^{\infty} a_2 r_2^n$, we can solve $|r_2| < 1$.

Even More Ex. 4:

$$|r_2| < 1$$

$$|2x| < 1$$

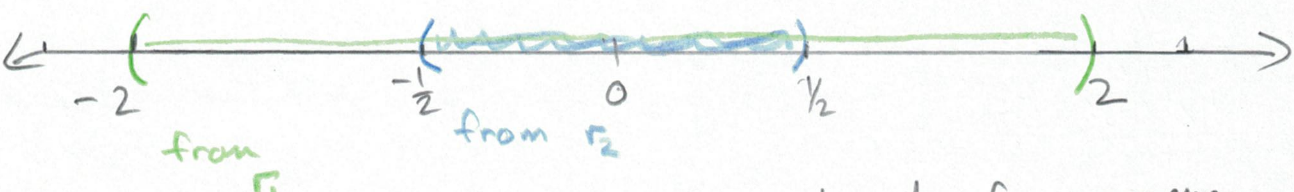
$$\tfrac{1}{2} \cdot |2x| < \tfrac{1}{2} \cdot 1$$

$$|x| < \tfrac{1}{2}$$

$$-\tfrac{1}{2} < x < \tfrac{1}{2}.$$

So, the interval of convergence for $\sum_{n=0}^{\infty} 2(2x)^n$ is $\left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$. $R_2 = \tfrac{1}{2}$

$R_2 = \tfrac{1}{2}$ $R_1 = 2$



Taking the intersection of these two intervals of convergence, we get $(-2, 2) \cap \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$ as the interval of convergence, or domain of $\sum_{n=0}^{\infty} \left[\frac{3(-1)^n x^n}{2^{n+1}} + 2^{n+1}\right] x^n$ as a power series representation.

Zoinks!

"sometimes, we take a step back to go forward."

Ex. 5: Use the geometric series sum formula to represent $f(x) = \ln(1-x^2)$ as a power series centered at $c=0$, and find the interval of convergence (domain) of this power series function. How does $\ln(1-x^2)$ look like $\frac{a}{1-r}$?

Consider differentiating: $\frac{d}{dx}[\ln(1-x^2)] = \left(\frac{1}{1-x^2}\right) \cdot \frac{d}{dx}[1-x^2]$

$$f'(x) = \frac{-2x}{1-x^2}$$

Well, $\frac{1}{1-x^2}$ looks like $\frac{a}{1-r}$, if $a=1$ and $r=x^2$, but we'll need to "adjust" our resulting power series using multiplication and integration to balance the fact that we used differentiation on $\ln(1-x^2)$.

Let's see how this works.

We can see $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (1)(x^2)^n$

$$\boxed{= \sum_{n=0}^{\infty} x^{2n}}$$

We can find the interval of convergence by solving $|r|<1$.

We have $|x^2| < 1$

$x^2 < 1$

$x^2 = 1$

Either $x=-1$, or $x=1$

Test $x=-2$ | Test $x=0$ | Test $x=2$

-1 | 0 | 1

False | True | False

$(-2)^2 < 1$ | $(0)^2 < 1$ | $(2)^2 < 1$

$4 < 1$ | $0 < 1$ | $4 < 1$

So, the interval of convergence is $(-1,1)$ for $f(x) = \sum_{n=0}^{\infty} x^{2n}$.

How do we get back to $\ln(1-x^2)$?

Consider $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$

$$\frac{x}{1} \cdot \left(\frac{1}{1-x^2}\right) = x \cdot \sum_{n=0}^{\infty} x^{2n}$$

$$\frac{x}{1-x^2} = \sum_{n=0}^{\infty} x^{2n+1}$$

$$\int \left(\frac{x}{1-x^2}\right) dx = \int \left[\sum_{n=0}^{\infty} x^{2n+1}\right] dx$$

More Ex. 5:

$$\int \frac{x}{1-x^2}\,dx = C + \sum_{n=0}^{\infty} \frac{x^{2n+2}}{2n+2}$$

Let $u=1-x^2$

$\frac{du}{dx} = -2x$

$du = \frac{du}{dx} \cdot dx$

$du = -2x\,dx$

$\frac{du}{-2x} = dx$

$$\int \frac{x}{u}\left(\frac{du}{-2x}\right) = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{2n+2} + C$$

$$-\frac{1}{2}\int \frac{1}{u}\,du = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{2n+2} + C$$

$$-\frac{1}{2}\ln|u| = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{2n+2} + C$$

$$-\frac{1}{2}\ln|1-x^2| = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{2n+2} + C$$

Can we solve for C? By Theorem 9.21, we know $\sum_{n=0}^{\infty} \frac{x^{2n+2}}{2n+2} + C$ converges on $(-1,1)$, so we can use $x=0$ to solve for C.

$$-\frac{1}{2}\ln|1-(0)^2| = \sum_{n=0}^{\infty} \frac{(0)^{2n+2}}{2n+2} + C$$

$$-\frac{1}{2}\ln(1) = 0 + C$$

$$0 = C$$

Therefore, $-\frac{1}{2}\ln(1-x^2) = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{2n+2}$ on $(-1,1)$.

$$\frac{-2}{1}\cdot\left[-\frac{1}{2}\ln(1-x^2)\right] = \frac{-2}{1}\cdot\sum_{n=0}^{\infty} \frac{x^{2n+2}}{2(n+1)}$$

$$f(x) = \ln(1-x^2) = -\sum_{n=0}^{\infty} \frac{x^{2n+2}}{n+1}$$

$$f(x) = -\sum_{n=0}^{\infty} \frac{x^{2(n+1)}}{n+1}$$

Re-write the index, and use $k=n+1$. If $n=0$, then $k=1$.

$$f(x) = -\sum_{k=1}^{\infty} \frac{x^{2k}}{k}$$

OR

$$\ln(1-x^2) = f(x) = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$$

which converges on $(-1,1)$.

Still More Ex. 5:

We have two issues to consider. First the domain of of the function, $f(x) = \ln(1-x^2)$ is $(-1,1)$, since $\ln(0)$ is undefined. However, we should verify that our power series representation does not converge at $x=-1$, or $x=1$.

check: $x=-1$, $-\sum_{n=1}^{\infty} \frac{x^{2n}}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$.

This series is a divergent harmonic series.

check: $x=1$, $-\sum_{n=1}^{\infty} \frac{x^{2n}}{n} = -\sum_{n=1}^{\infty} \frac{(1)^{2n}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$.

This series is a divergent harmonic series.

"We take a step back To go forward."

Ex. 6: Use the geometric series sum formula to represent $f(x) = \arctan(x)$ as a power series centered at $c = 0$, and find the interval of convergence (domain) of this power series function.

How does $\arctan(x)$ look like $\frac{a}{1-r}$?

Consider differentiating: $\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2} = f(x)$

Well, $\frac{1}{1+x^2}$ looks like $\frac{a}{1-r}$, if $a=1$ and $r=-x^2$, but we'll need to "adjust" our resulting power series using integration to balance the fact we used differentiation on $\arctan(x)$. Let's see how this works.

We can see $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$

$$= \sum_{n=0}^{\infty} (1)(-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x^2)^n$$

$$\boxed{\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}}$$

We can find the interval of convergence by solving $|r|<1$.

We have $|-x^2|<1$

$$x^2<1$$

$$-1<x<1.$$

So, the interval of convergence is $(-1,1)$ for $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

Now, we'll using integration to get back to $f(x) = \arctan(x)$.

Consider $f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$$\int \left(\frac{1}{1+x^2}\right) dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx$$

$$\arctan(x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Use $x=0$ to find C. $\arctan(0) = C + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (0)^{2n+1}}{2n+1}$

$$0 = C + 0$$

$$0 = C$$

More Ex. 6:

According to Theorem 9.21, $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ converges on $(-1,1)$ as well, but we need to check the endpoints of this interval.

Check: $x=-1$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+1}$ ← Always odd

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (-1)}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

This series converges according to the Alternating Series Test.
Let $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^{n+1} a_n$, with $a_n = \frac{1}{2n+1}$. We can see that $\frac{1}{2n+1} > 0$ as a ratio of positive numbers is positive. (RoPNiP) So, $a_n > 0$.

(I) Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n+1} = 0$ ✓

(II) We know $2n+1 \le 2n+3$ for all $n \ge 1$.

$$\frac{1}{(2n+1)(2n+3)}\left[\frac{2n+1}{1}\right] \le \frac{1}{(2n+1)(2n+3)}\cdot\left[\frac{2n+3}{1}\right]$$

$$\frac{1}{2n+3} \le \frac{1}{2n+1}$$

$$a_{n+1} \le a_n$$

Therefore, $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges.

Check: $x=1$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 1}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

This series converges according to the Alternating Series Test, with $a_n = \frac{1}{2n+1}$, by letting $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} (-1)^n a_n$.

Still More Ex. 6:

This means that $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ on $[-1, 1]$.

By the way, if $x=1$, then $\arctan(1) = \theta$, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\tan[\arctan(1)] = \tan(\theta)$$
$$1 = \tan(\theta)$$
$$\theta = \frac{\pi}{4}$$

Therefore, $\arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1}$

and

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$
$$4 \cdot \frac{\pi}{4} = 4 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$
$$\underline{\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}}$$

← An interesting statement about computing the value of π to any desired accuracy.